Analysis of Planar Waveguides with the Method of Lines and Absorbing Boundary Conditions

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Abstract—**The method of lines is extended to analyze planar waveguides with open boundaries. The efficiency of using absorbing boundary conditions is demonstrated by the calculation of the effective permittivity of a single microstrip line.**

I. Introduction

To ANALYZE planar waveguide structures numerically, they must be enclosed by walls to limit the area of discretization. The use of electric or magnetic walls produces errors, since the corresponding tangential field components are set to zero. Radiation effects cannot be taken into account. To overcome this difficulty the free space can be simulated at an arbitrarily fixed position by using absorbing boundary conditions [1]. In the following these conditions are adapted to the method of lines [4]. An additional advantage of the method of lines is, that only two of these boundaries in the direction of discretization are necessary since the remaining system of ordinary differential equations is solved analytically.

II. Analysis

A. Factorization of the Helmholtz Operator

Assuming a wave propagation $\exp(-jk_z z)$ in z-direction, the wave equation for the two independent field components can be written as

$$L\psi = \left(D_{\bar{x}}^2 + D_{\bar{y}}^2 + \epsilon_d\right)\psi = 0 \qquad (1)$$

with

$$D_{\bar{x}}^2 \equiv \frac{\partial^2}{\partial \bar{x}^2}, \qquad D_{\bar{y}}^2 \equiv \frac{\partial^2}{\partial \bar{y}^2}, \qquad \epsilon_d = \epsilon_r - \epsilon_{re}$$

and the normalizations

$$\bar{x} = x \cdot k_0, \qquad \bar{y} = y \cdot k_0, \qquad \epsilon_{re} = \frac{k_z^2}{k_0^2},$$

where ψ stands for e_z or h_z, respectively. The Helmholtz operator L can be factored so that

$$L\psi = L^+ L^- \psi = 0 \qquad (2)$$

with

$$L^{\pm} = D_{\bar{x}} \pm j\sqrt{\epsilon_d}\,\sqrt{1+S^2} \quad S^2 = \frac{D_{\bar{y}}^2}{\epsilon_d}.$$

Since in general ϵ_d will be complex, the branch cut of the square root function must be chosen so that the plus sign is related to waves traveling in the x-direction, the minus sign to those traveling in the $-x$-direction. Hence the condition for outgoing waves is.

$$\Re\sqrt{\epsilon_d} > 0$$

while

$$\Im\sqrt{\epsilon_d} < 0$$

holds for exponentially decaying waves if $\Re\sqrt{\epsilon_d} = 0$ and no propagation takes place. The conditions

$$L^- = 0 \qquad (3)$$

at the left and

$$L^+ = 0 \qquad (4)$$

at the right boundary exclude solutions of (1) which are related to waves traveling backward. This is in accordance with Sommerfeld's condition of radiation [3]. In this way, (3) and (4) simulate exactly absorbing boundaries for waves incident at any angle from the interior of the enclosed structure.

B. Discretization

To use these boundary conditions in connection with the method of lines, the radical is approximated by a polynomial of the form

$$\sqrt{1+S^2} \approx p_0 + p_2 S^2 \qquad (5)$$

on the interval $[-1,1]$. The choice of the coefficients p_0 and p_2 depends on the method of interpolation and determines the two angles of exact absorption [2]. To keep the advantageous procedure with shifted line systems [4] the boundary conditions

$$\left(D_{\bar{y}}^2 \mp j\frac{\sqrt{\epsilon_d}}{p_2}D_{\bar{x}} + \frac{p_0}{p_2}\epsilon_d\right)e_z = 0 \qquad (6)$$

$$\left(D_{\bar{y}}^2 \mp j\frac{\sqrt{\epsilon_d}}{p_2}D_{\bar{x}} + \frac{p_0}{p_2}\epsilon_d\right)\frac{\partial h_z}{\partial x} = 0 \qquad (7)$$

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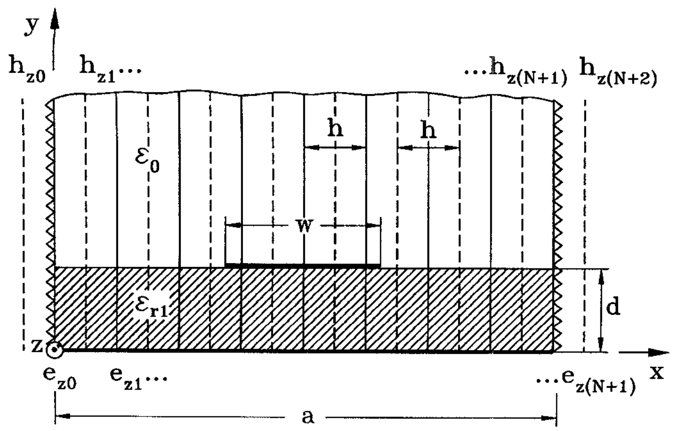

Fig. 1. Cross section of investigated microstrip waveguide with discretization lines.

are formulated. The physical consequence of the second equation on the dashed lines (Fig. 1) is the validity of the radiation condition for the tangential component of the electric field strength e_y.

The discretization of (1) and (6) leads to the difference equations

$$D_{\bar{y}}^2 e_{z1} + \frac{jn_d}{2p_2} \cdot \frac{e_{z2} - e_{z0}}{\bar{h}^2} + \epsilon_d \frac{p_0}{p_2} e_{z1} = 0, \tag{8}$$

$$D_{\bar{y}}^2 e_{z1} + \frac{e_{z0} - 2e_{z1} + e_{z2}}{\bar{h}^2} + \epsilon_d e_{z1} = 0, \tag{9}$$

$$\vdots$$

$$D_{\bar{y}}^2 e_{zN} + \frac{e_{z(N-1)} - 2e_{zN} + e_{z(N+1)}}{\bar{h}^2} + \epsilon_d e_{zN} = 0, \tag{10}$$

$$D_{\bar{y}}^2 e_{zN} - \frac{jn_d}{2p_2} \cdot \frac{e_{z(N-1)} - e_{z(N+1)}}{\bar{h}^2} + \epsilon_d \frac{p_0}{p_2} e_{zN} = 0, \tag{11}$$

with $\bar{h} = h \cdot k_0, n_d = \bar{h}\sqrt{\epsilon_d}$. Similar expressions can be found for the corresponding problem on the dashed line system (7). Combining (8) and (9) as well as (10) and (11), e_{z0} and $e_{z(N+1)}$ are eliminated and the difference equation system can be written as

$$\left(D_{\bar{y}}^2 - \bar{h}^{-2}\boldsymbol{P} + \epsilon_d \boldsymbol{I}\right)\boldsymbol{\psi} = \boldsymbol{0}, \tag{12}$$

where $\boldsymbol{\psi}$ is either $\boldsymbol{E}_z$ or $\boldsymbol{H}_z$, which are the discretized field components combined to a vector. $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{P}$ the corresponding difference matrix

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & p_{13} & p_{12} & p_{11} \end{bmatrix} \tag{13}$$

with

$$p_{11} = 2 + a, \qquad p_{12} = -1 - b, \qquad p_{13} = 0,$$

for $\boldsymbol{\psi} = \boldsymbol{E}_z$ and

$$p_{11} = 1 + a, \qquad p_{12} = -1 - b - a, \qquad p_{13} = b,$$

for $\boldsymbol{\psi} = \boldsymbol{H}_z$. Moreover the abbreviations

$$a = -2\frac{2p_2 + (p_0 - p_2)n_d^2}{2p_2 + jn_d}, \qquad b = -\frac{2p_2 - jn_d}{2p_2 + jn_d}$$

have been used. Notice that $a = b = 0$ yields the appropriate difference matrices for electric walls.

Introducing the difference operators

$$\bar{h}\frac{\partial e_z}{\partial x} \to \boldsymbol{D}_e \boldsymbol{E}_z \tag{14}$$

$$\bar{h}\frac{\partial h_z}{\partial x} \to -\boldsymbol{D}_h \boldsymbol{H}_z, \tag{15}$$

with

$$\boldsymbol{D}_e = \begin{bmatrix} 1 + a & -b & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & b & -1 - a \end{bmatrix}$$

$$\boldsymbol{D}_h = \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ & & 1 & -1 \end{bmatrix}, \tag{16}$$

the matrix $\boldsymbol{P}$ can be represented as the product

$$\boldsymbol{P}_e = \boldsymbol{D}_h \boldsymbol{D}_e \qquad \boldsymbol{P}_h = \boldsymbol{D}_e \boldsymbol{D}_h. \tag{17}$$

If $\boldsymbol{P}_e$ is of the order N then $\boldsymbol{P}_h$ is of the order $N + 1$. In order to solve (12), a transformation to principle axes is made

$$\boldsymbol{\psi} = \boldsymbol{T}\bar{\boldsymbol{\psi}} \tag{18}$$

with

$$\boldsymbol{T}^{-1}\boldsymbol{P}\boldsymbol{T} = \boldsymbol{\lambda}^2. \tag{19}$$

The eigenvalues $\boldsymbol{\lambda}_e^2$ and eigenvectors $\boldsymbol{T}_e$ are calculated numerically. Taking (17) and (19) into account, the solutions of the eigenvalue problem with $\boldsymbol{P}_h$ are given by

$$\boldsymbol{\lambda}_h^2 = \begin{bmatrix} 0 & \\ & \boldsymbol{\lambda}_e^2 \end{bmatrix} \boldsymbol{T}_h = \begin{bmatrix} \frac{1}{\sqrt{N+1}} & \\ \vdots & \boldsymbol{D}_e \boldsymbol{T}_e \boldsymbol{\lambda}_e^{-1} \\ \frac{1}{\sqrt{N+1}} & \end{bmatrix}, \tag{20}$$

since it can be shown that one eigenvalue is zero. With the definition of the normalized quasi diagonal (to prove this fact use (20)) matrices

$$\bar{\boldsymbol{d}}_e = \boldsymbol{T}_h^{-1}\boldsymbol{D}_e\boldsymbol{T}_e(k_0 h)^{-1} = \begin{bmatrix} \cdots 0 \cdots \\ \bar{\boldsymbol{\lambda}}_e \end{bmatrix}$$

$$\bar{\boldsymbol{d}}_h = \boldsymbol{T}_e^{-1}\boldsymbol{D}_h\boldsymbol{T}_h(k_0 h)^{-1} = \begin{bmatrix} \vdots & \\ 0 & \bar{\boldsymbol{\lambda}}_e \\ \vdots & \end{bmatrix}, \tag{21}$$

the following analysis is done in the same way as described in [4], but it is important to notice, that the eigenvalues and

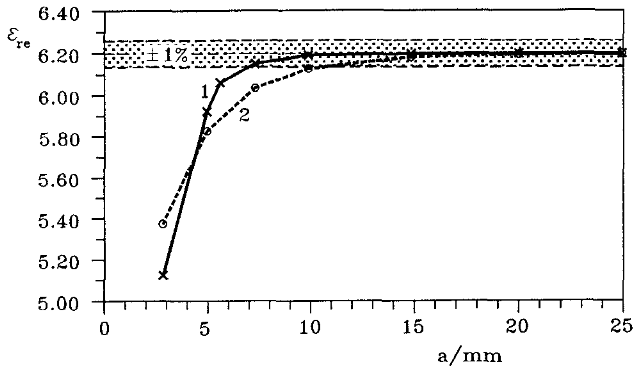

Fig. 2. Convergence of ϵ_{re} with increasing distance of boundaries ($w = 1$ mm, $d = 1$ mm, $\epsilon_{r1} = 8.875$, $f = 5$ GHz, 9 lines on the strip). 1: Absorbing boundaries; 2: metallic walls.

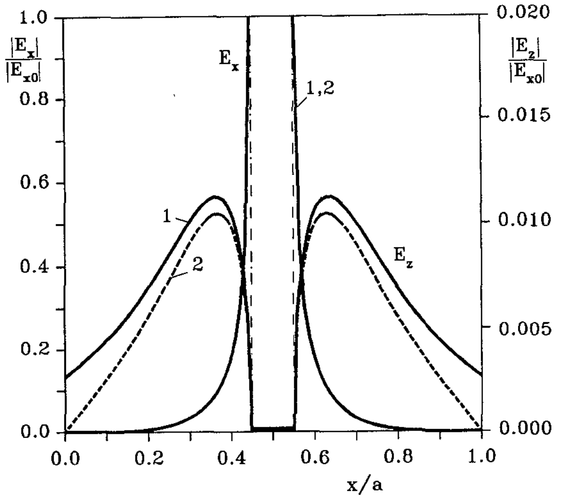

Fig. 3. Electric field components $|E_x|$ and $|E_z|$ in the plane of the strip, $y = d$, ($a = 9.88$ mm, remaining dimension as in Fig. 2). $|E_{x0}|$ is the field component on the line at $x = 0.5(a - w - 0.5h)$. 1: Absorbing boundaries; 2: metallic walls.

the transformation matrix must be calculated separately for each layer. The zero of the determinant of the reduced system matrix, the effective permittivity ϵ_{re}, lies in the complex plane. The imaginary part of ϵ_{re} is very small and due to the fact that the energy outside of the lateral walls is absorbed. With increasing distance a, refer to Fig. 1, the imaginary part of ϵ_{re} decreases to zero.

III. Results

To show the efficiency of absorbing boundary conditions, the effective permittivity $\epsilon'_{re} = \Re\, \epsilon_{re}$ of a single microstrip line has been computed (Fig. 2). The coefficients of the polynomial have been set to

$$p_0 = 1, \qquad p_2 = \frac{1}{2}, \tag{22}$$

corresponding to a Taylor series approximation of the radical function. With increasing distance a, refer to Fig. 1, the value of S^2 decreases to zero. Therefore the approximation of the radical in (5) fits better and better. It is evident, that metallic walls must be more distant from the strip to achieve good results. The modulus of the electric field strength in the plane of the strip is shown in Fig. 3. The curves (1) and (2) of the field component E_x normal to the boundary are nearly identical, whereas the tangential field component E_z is nonzero at the position of the absorbing boundary.

IV. Conclusion

By the use of absorbing boundary conditions, the method of lines has been extended to analyse laterally open planar waveguides. The results show that the boundaries may be placed closer to the guiding structure whereby the discretized area and the corresponding matrices become smaller.

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